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A CONDITION NUMBER FOR LINEAR INEQUALITIES AND LINEAR PROGRAMS. (U)

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A CONDITION NUMBER FOR
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AND LINEAR PROGRAMS

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O. L. Mangasarian

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ABSTRACT

A new explicit bound is given for the ratio of the absolute error in an approximate solution of a system of linear inequalities and equalities to the absolute residual. This bound generalizes the concept of a norm of the inverse of a nonsingular matrix. With this bound a condition number is defined for a system of linear inequalities and equalities and for linear programs. The condition number gives a bound on the ratio of the relative error of an approximate solution to the relative residual. In the case of a strongly stable system of linear inequalities and equalities the condition number can be computed by means of a single linear program.

AMS (MOS) Subject Classifications: 15A39, 90C05, 65F35

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SIGNIFICANCE AND EXPLANATION

In solving a system of nonsingular linear equations the condition number gives a useful bound on the ratio of the relative error of an approximate solution to the relative residual. We extend this bound to the important cases of linear equations and inequalities and of linear programs which one commonly encounters in operations research.

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A CONDITION NUMBER FOR LINEAR INEQUALITIES
AND LINEAR PROGRAMS

O. L. MANGASARIAN

1. Introduction

A well known result of linear algebra [1, 8] states that for the system of linear equation

$$Ex = g \quad (1)$$

where E is a given $n \times n$ real nonsingular matrix and g is a given nonzero vector in the n -dimensional real Euclidean space R^n , the following bound for the ratio of the relative error of an approximate solution to the relative residual holds for any fixed norm:

$$\frac{\|x - \bar{x}\|}{\|\bar{x}\|} \leq \|E^{-1}\| \|E\| \frac{\|Ex - g\|}{\|g\|} \quad (2)$$

Here \bar{x} is the exact solution, that is $E\bar{x} = g$, whereas x is an approximate solution with residual $Ex - g$. The quantity $\|E^{-1}\| \|E\|$, which depends on the norm employed, is called the condition number of the matrix E . The condition number is a useful numerical constant which measures how badly the relative error could behave in terms of the relative residual. The purpose of this work is to define a similar quantity for the system of linear inequalities and equalities

$$Ax \leq b, \quad Cx = d \quad (3)$$

where A and C are given $m \times n$ and $k \times n$ real matrices respectively, b is a given vector in the m -dimensional real Euclidean space R^m and d is a given vector in R^k .

Our results depend on a theorem due to Hoffman [4] which gives a bound on the ratio of the absolute error in an approximate solution of a system of linear inequalities to the absolute value of the residual. We shall give here a new explicit expression for this bound by using a particularly simple proof that makes use of duality theory of linear programming [3]. The bound in Hoffman's theorem is a generalization of the idea of a norm of the inverse of a nonsingular matrix (Remark 2). For monotonic norms the product of this bound with the norm of the matrices involved generates what we have termed a condition number for matrices associated with a system of inequalities and equalities (Theorem 2 and Definition 1). For the case of a system of inequalities which satisfy a strong regularity condition we show how the condition number can be obtained by solving a single linear program (Theorem 3). In Section 3 we extend the concept of a condition number to a pair of dual linear programs.

We briefly describe now the notation and some of the basic concepts used in this work. For a vector x in the n -dimensional real Euclidean space R^n , $|x|$ and x_+ will denote the vectors in R^n with components $|x|_i = |x_i|$ and $(x_+)_i = \max\{x_i, 0\}$, $i = 1, 2, \dots, n$, respectively. For a norm $\|x\|_\beta$ on R^n , $\|x\|_\beta^*$ will denote the dual norm [5, 10] on R^n , that is

$$\|x\|_\beta^* = \max_{\|y\|_\beta = 1} x^T y, \text{ where the superscript } T \text{ denotes the transpose. The}$$

generalized Cauchy-Schwarz inequality $|x^T y| \leq \|x\|_\beta \|y\|_\beta^*$, for x and y in R^n , follows immediately from this definition of the dual norm. For $1 \leq p$,

$q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ the p -norm $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ and the q -norm are dual norms in R^n [8]. If $\|\cdot\|_\beta$ is a norm on R^n , we shall, with a slight abuse of notation, let $\|\cdot\|_\beta$ also denote the corresponding norm on R^m for $m \neq n$.

For an $m \times n$ real matrix A , A_i denotes the i th row and $A_{\cdot j}$ denotes the

jth column, while $\|A\|_\beta$ denotes the matrix norm [11] subordinate to the vector norm $\|\cdot\|_\beta$, that is $\|A\|_\beta = \max_{\|x\|_\beta=1} \|Ax\|_\beta$. The consistency condition

$\|Ax\|_\beta \leq \|A\|_\beta \|x\|_\beta$ follows immediately from this definition of a matrix norm. A vector of ones in any real Euclidean space will be denoted by e .

2. Condition Number for a System of Inequalities and Equalities

We begin by stating and proving a form of Hoffman's theorem [4] which is suitable for our purposes and which includes an explicit expression, not given before, for the value of the bound on the ratio of the error in an approximate solution to the residual. Robinson [9] has given a different but more complicated expression for this bound.

Theorem 1. Let A, b, C and d be as defined in (3), let $\|\cdot\|_\beta$ be a norm on R^{m+k} and let

$$S = \{x \mid Ax \leq b, Cx = d\} \quad (4)$$

be nonempty. Then for each x in R^n not in S there exists a point $p(x)$ in S such that

$$\|x - p(x)\|_\infty \leq \mu_{\beta\infty}(A, C) \cdot \|(Ax-b)_+, (Cx-d)\|_\beta \quad (5)$$

where

$$\mu_{\beta\infty}(A, C) := \text{maximum}_{u, v, w, y, \zeta} \left\{ \|w, y - e\zeta\|_\beta \mid \begin{array}{l} (u, v, w, y, \zeta) \text{ is a vertex of:} \\ u - v - A^T w - C^T y + C^T e \zeta = 0 \\ e^T u + e^T v = 1 \\ (u, v, w, y, \zeta) \geq 0 \end{array} \right\} \quad (6)$$

Remark 1.

The subscripts β and ∞ of μ refer to the β -norm and ∞ -norm used in (5). We shall give below in (9) a relation using other norms. The condition in (6) that (u, v, w, y, ζ) be a vertex is essential, for otherwise the maximum of (6) may not exist. We shall give below (Corollary 2 and Theorem 3) cases where the vertex requirement need not be imposed.

Proof of Theorem 1. Let x be any point in R^n not in S . Because the set S is nonempty there exists a point $p(x)$ in S which is closest to x in the ∞ -norm. For such a point, $(p(x), \|p(x) - x\|_\infty)$ constitute a solution of

the linear program

$$\min_{p, \delta} \{ \delta \mid -e\delta \leq p-x \leq e\delta, Ap \leq b, Cp = d \} . \quad (7)$$

The dual of (7) is

$$\max_{u, v, w, y, \zeta} \{ x^T(u-v) - b^T w - d^T(y - e\zeta) \mid u-v - A^T w - C^T y + C^T e\zeta = 0, e^T(u+v) = 1, (u, v, w, y, \zeta) \geq 0 \} \quad (8)$$

with a vertex solution $(u(x), v(x), w(x), y(x), \zeta(x))$ which satisfies

$$0 < \delta(x) := \|p(x) - x\|_{\infty} = x^T(u(x) - v(x)) - b^T w(x) - d^T(y(x) - e\zeta(x))$$

$$= w(x)^T(Ax-b) + (y(x) - e\zeta(x))^T(Cx-d)$$

$$\leq w(x)^T(Ax-b)_+ + (y(x) - e\zeta(x))^T(Cx-d)$$

$$(\text{Since } (Ax-b)_+ \geq Ax-b)$$

$$\leq \|w(x), (y(x) - e\zeta(x))\|_+ \cdot \|(Ax-b)_+, (Cx-d)\|_{\beta}$$

(By the generalized Cauchy-Schwarz inequality)

$$\leq \mu_{\beta\infty}(A, C) \|(Ax-b)_+, (Cx-d)\|_{\beta}$$

□

Remark 2. For the case when $Ax \leq b$ is absent from (3) and (4), C is a nonsingular $n \times n$ matrix and $\beta = \infty$, definition (6) degenerates to

$$\mu_{\infty}(\phi, C) = \|C^{-1}\|_{\infty}.$$

Proof of Remark 2.

$$\begin{aligned} \mu_{\infty}(\phi, C) &= \max_{u, v, y, z} \left\{ \|y - ez\|_1 \left| \begin{array}{l} (u, v, y, z) \text{ is a vertex of:} \\ y - ez = (C^T)^{-1} (u - v) \\ e^T u + e^T v = 1 \\ (u, v, y, z) \geq 0 \end{array} \right. \right\} \\ &= \max_{u, v, y, z} \left\{ \|y - ez\|_1 \left| \begin{array}{l} y - ez = (C^T)^{-1} (u - v) \\ e^T u + e^T v = 1 \\ (u, v, y, z) \geq 0 \end{array} \right. \right\} \end{aligned}$$

(Because the convex function $\|y - ez\|_1$ attains its maximum at a vertex [10, Corollary 32.3.4])

$$\begin{aligned} &= \max_{u, v} \left\{ \|(C^T)^{-1} (u - v)\|_1 \left| \begin{array}{l} e^T u + e^T v = 1 \\ (u, v) \geq 0 \end{array} \right. \right\} \\ &= \max_{1 \leq i \leq n} \left\| \pm ((C^T)^{-1})_{\cdot i} \right\|_1 \end{aligned}$$

(Since $\|(C^T)^{-1} (u - v)\|_1$ is convex and attains its maximum at one of the vertices of the simplex $\{(u, v) | e^T u + e^T v = 1, (u, v) \geq 0\}$. These are the vertices of the unit cube $\{(u, v) | 0 \leq (u, v) \leq e\}$ lying along the coordinate axes and excluding the origin.)

$$= \|C^{-1}\|_{\infty}$$

□

Theorem 1 can be easily stated for norms other than the ∞ -norm as follows.

Corollary 1. Let the assumptions of Theorem 1 hold. Then for each x not in S there exists $p(x)$ in S such that

$$\|x - p(x)\|_{\gamma} \leq \mu_{\beta\gamma}(A, C) \cdot \|(Ax-b)_+, (Cx-d)\|_{\beta} \quad (9)$$

where

$$\mu_{\beta\gamma}(A, C) = \alpha_{\gamma\infty} \mu_{\beta\infty}(A, C) \quad (10)$$

and $\alpha_{\gamma\infty}$ is the positive number relating the γ -norm and ∞ -norm by

$$\|z\|_{\gamma} \leq \alpha_{\gamma\infty} \|z\|_{\infty} \text{ for all } z \text{ in } R^n.$$

Before deriving the condition-number result for the system (3) we need a couple of simple lemmas. We recall [5, 8] that a monotonic norm on R^n is any norm $\|\cdot\|$ on R^n such that for a, b in R^n , $\|a\| \leq \|b\|$ whenever

$|a| \leq |b|$. The q -norm, $\|a\|_q = \left(\sum_{i=1}^n |a_i|^q \right)^{\frac{1}{q}}$ for $q \geq 1$ is a monotonic norm [8].

Lemma 1. Let a and b be real numbers. Then $a \leq b$ implies $(a)_+ \leq |b|$.

Proof. Let $a \leq b$. Then

$$|b| - (a)_+ = 2(b)_+ - b - (a)_+ = ((b)_+ - b) + ((b)_+ - (a)_+) \geq 0.$$

□

Lemma 2. Let $\|\cdot\|_{\beta}$ be a monotonic norm on R^{m+k} , let a, b be in R^m and let c, d be in R^k . Then $a \leq b$ and $c = d$ imply that

$$\|(a)_+, c\|_{\beta} \leq \|b, d\|_{\beta}.$$

Proof. Let $a \leq b$ and $c = d$. By Lemma 1 we have that $(a)_+ \leq |b|$, and we also have that $|c| = |d|$. Hence by the monotonicity of $\|\cdot\|_{\beta}$ we have that

$$\|(a)_+, c\|_{\beta} \leq \|b, d\|_{\beta}.$$

□

Lemma 3. Let A, b, C and d be as defined in (3) and let $\|\cdot\|_{\beta}$ be a monotonic norm on R^{m+k} . Then $\|(-b)_+, d\|_{\beta} \leq \|Ax, Cx\|_{\beta}$ for any x in S .

Proof. Just apply Lemma 2 to $-b \leq -Ax$ and $d = Cx$.

□

We are now ready to establish our condition-number result for the system (3).

Theorem 2. Let A, b, C, d and S be as defined in (3) and (4), let $\|\cdot\|_\beta$ be a monotonic norm on \mathbb{R}^{m+k} and let $((-b)_+, d) \neq 0$. Then there exists a number $\mu_{\beta\beta}(A, C)$ defined by (10) and (6) such that for each x in \mathbb{R}^n not in S there exists a $p(x)$ in S such that

$$\frac{\|x - p(x)\|_\beta}{\|p(x)\|_\beta} \leq \mu_{\beta\beta}(A, C) \left\| \begin{matrix} A \\ C \end{matrix} \right\|_\beta \frac{\|(Ax-b)_+, (Cx-d)\|_\beta}{\|(-b)_+, d\|_\beta}. \quad (11)$$

Proof. Let x be any point in \mathbb{R}^n not in S . By Corollary 1 there exists a point $p(x)$ in S satisfying (9). Obviously $p(x) \neq 0$, because $((-b)_+, d) \neq 0$, and

$$\begin{aligned} \frac{\|x - p(x)\|_\beta}{\|p(x)\|_\beta} &\leq \mu_{\beta\beta}(A, C) \frac{\|(-b)_+, d\|_\beta}{\|p(x)\|_\beta} \frac{\|(Ax-b)_+, (Cx-d)\|_\beta}{\|(-b)_+, d\|_\beta} \\ &\leq \mu_{\beta\beta}(A, C) \frac{\|Ap(x), Cp(x)\|_\beta}{\|p(x)\|_\beta} \frac{\|(Ax-b)_+, (Cx-d)\|_\beta}{\|(-b)_+, d\|_\beta} \end{aligned}$$

(By Lemma 3)

$$\leq \mu_{\beta\beta}(A, C) \left\| \begin{matrix} A \\ C \end{matrix} \right\|_\beta \frac{\|(Ax-b)_+, (Cx-d)\|_\beta}{\|(-b)_+, d\|_\beta}$$

□

It is evident from (11) that the following is an appropriate definition for a condition number of (A, C) relative to the inequalities and equalities (3).

Definition 1. Let A, b, C and d be as defined in (3), and let $\|\cdot\|_\beta$ be a monotonic norm on \mathbb{R}^{m+k} . The condition number $\nu_\beta(A, C)$ of the matrices (A, C) relative to the system of inequalities and equalities of (3) is defined by

$$v_{\beta}(A,C) := \mu_{\beta\beta}(A,C) \left\| \begin{pmatrix} A \\ C \end{pmatrix} \right\|_{\beta} \quad (12)$$

where $\mu_{\beta\beta}(A,C)$ is defined by (10) and (6).

In view of Remark 2 we have that $v_{\infty}(\phi, C) = \|C^{-1}\|_{\infty} \|C\|_{\infty}$ when C is nonsingular and A is not present.

We note that the computation of the condition number $v_{\beta}(A,C)$ of (12) depends on computing $\mu_{\beta\infty}(A,C)$ of (6) which is a problem of maximizing of a convex function on the vertices of a polyhedral set. This is a difficult problem because it may have many local maxima. This problem however can be reduced to a maximization problem on the entire polyhedral set rather than on its vertices, if the objective function is bounded above on the entire polyhedral set. In particular we have the following:

Corollary 2. Let A and C be $m \times n$ and $k \times n$ real matrices respectively and let $\|\cdot\|_{\beta}$ be a norm on R^{m+k} . If

$\{x | Ax < 0, Cx = 0\} \neq \phi$ and the rows of C are linearly independent (13)

then the constant $\mu_{\beta\infty}(A,C)$ of (6) is also given by

$$\mu_{\beta\infty}(A,C) = \text{maximum}_{u,v,w,z} \left\{ \|w, z\|_{\beta} \mid \begin{array}{l} u - v - A^T w - C^T z = 0 \\ e^T u + e^T v = 1 \\ (u, v, w) \geq 0 \end{array} \right\} \quad (14)$$

Proof. Note that problem (14) is obtained from (6) by dropping the vertex condition from (6) and defining $z = y - e\zeta$. We first show that the feasible region of (14) is bounded under assumption (13). Suppose not, then there exists a sequence $\{(u^i, v^i, w^i, z^i)\}$ with $\|w^i, z^i\| \rightarrow \infty$ for some norm, $\{(u^i, v^i)\}$ remaining bounded and

$$\frac{u^i - v^i - A^T w^i - C^T z^i}{\|w^i, z^i\|} = 0, \quad \frac{w^i}{\|w^i, z^i\|} \geq 0.$$

Hence by the Bolzano-Weierstrass theorem there exists (\bar{w}, \bar{z}) such that

$$A^T \bar{w} + C^T \bar{z} = 0, \bar{w} \geq 0, (\bar{w}, \bar{z}) \neq 0 \quad (15)$$

which contradicts the linear independence of the rows of C if $\bar{w} = 0$ and contradicts the nonemptiness of $\{x | Ax < 0, Cx = 0\}$ if $\bar{w} \neq 0$ because then $0 > x^T A^T \bar{w} + x^T C^T \bar{z} = 0$. Hence the feasible region of (14) is compact and the continuous objective function $\|w, z\|_{\beta}$ attains a maximum on it. Since problem (14) is equivalent to problem (6) without the vertex requirement, then problem (6) without the vertex condition must have a solution. Since the objective function of (6) is concave and the feasible region is a polytope not containing straight lines going to infinity in both directions, problem (6) without the vertex condition must have a vertex solution [10, Corollary 32.3.4]. Hence under assumption (13) the vertex condition can be dropped from (6) which results in (14).

□

Remark 3. It can be shown [7] that condition (13) is equivalent to the strong regularity or strong stability condition that

$$Ax \leq b, Cx = d \text{ has solution } x \text{ for each } (b, d) \in R^{m+k}. \quad (16)$$

For the case when there are no equalities present in (3), $\beta = \infty$ and $\beta^* = 1$, problem (14) becomes a simple linear program and the condition number of the matrix A with respect to the inequalities $Ax \leq b$ can be determined by solving a single linear program. In particular we have the following.

Theorem 3. Let A be an $m \times n$ real matrix, let b be a vector in R^m , let $S = \{x | Ax \leq b\}$ and let $\{x | Ax < 0\} \neq \emptyset$. Then for each x in R^n not in S there exists a point $p(x)$ in S such that

$$\|x - p(x)\|_{\infty} \leq \mu_{\infty}(A) \|Ax - b\|_{+} \quad (17)$$

where

$$\mu_{\infty}(A) = \max_{u,v,w} \left\{ e^T w \left| \begin{array}{l} u - v - A^T w = 0, \quad e^T u + e^T v = 1 \\ u, v, w \geq 0 \end{array} \right. \right\}, \quad (18)$$

Furthermore if $(-b)_+ \neq 0$

$$\frac{\|x - p(x)\|_{\infty}}{\|p(x)\|_{\infty}} \leq \mu_{\infty}(A) \|A\|_{\infty} \frac{\|(Ax - b)_+\|_{\infty}}{\|(-b)_+\|_{\infty}}. \quad (19)$$

The number $\mu_{\infty}(A) \|A\|_{\infty}$ is termed as the condition number of the matrix A relative to the inequalities $Ax \leq b$.

3. Condition Number of a Linear Program

We now apply the results of the previous section to obtain some corresponding results for a linear programming problem. Consider the linear program

$$\text{Minimize}_{z \in R^{n_1}} \{g^T z \mid Hz \geq h, z \geq 0\} \quad (20)$$

when g and h are given vectors in R^{n_1} and R^{n_2} respectively and H is a given $n_2 \times n_1$ real matrix. Associated with the above linear program and its dual are the following necessary and sufficient optimality conditions [3, 2]

$$-Mx - q \leq 0, -x \leq 0, q^T x \leq 0 \quad (21a)$$

where

$$M = \begin{bmatrix} 0 & -H^T \\ H & 0 \end{bmatrix}, \quad q = \begin{bmatrix} g \\ -h \end{bmatrix}, \quad x = \begin{bmatrix} z \\ s \end{bmatrix}. \quad (21b)$$

We have the following counterparts for the theorems of Section 2.

Theorem 4. Consider the linear program (20) and its associated optimality conditions (21). Let

$$S = \{x \mid -Mx \leq q, -x \leq 0, q^T x \leq 0\} \neq \emptyset \quad (22)$$

and let $\|\cdot\|_\beta$ be a norm on R^m where $m = 2(m_1 + n_1) + 1$. Then for each x not in S there exists a point $p(x)$ in S such that

$$\|x - p(x)\|_\infty \leq \mu_\beta \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \cdot \|(-Mx - q, -x, q^T x)_+\|_\beta \quad (23)$$

where

$$\mu_{\beta\infty} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} := \text{maximum}_{u,v,w_1,w_2,w_3} \left\{ \|w_1, w_2, w_3\|_{\beta}^* \mid \begin{array}{l} (u,v,w_1,w_2,w_3) \text{ is a} \\ \text{vertex of:} \\ u-v+M^T w_1 + w_2 - q^T w_3 = 0 \\ e^T u + e^T v = 1 \\ (u,v,w_1,w_2,w_3) \geq 0 \end{array} \right\} \quad (24)$$

If the ∞ -norm is replaced by the γ -norm in the above, then

$$\mu_{\beta\infty} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \text{ must be replaced by}$$

$$\mu_{\beta\gamma} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} = \alpha_{\gamma\infty} \mu_{\beta\infty} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \quad (25)$$

where $\alpha_{\gamma\infty}$ is the positive number relating the γ -norms and the ∞ -norms by $\|z\|_{\gamma} \leq \alpha_{\gamma\infty} \|z\|_{\infty}$ for all z in $R^{n_1+m_1}$.

Theorem 5. (Condition number of a linear program) Let the assumptions of Theorem 4 hold, let $\|\cdot\|_{\beta}$ be a monotonic norm on $R^{2(m_1+n_1)+1}$ and let

$$q \neq 0. \text{ Then there exists a number } \mu_{\beta\beta} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \text{ defined by (25) and (24)}$$

such that for each x not in S there exists a $p(x)$ in S such that

$$\frac{\|x - p(x)\|_{\beta}}{\|p(x)\|_{\beta}} \leq \mu_{\beta\beta} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \left\| \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \right\|_{\beta} \frac{\|(-Mx - q, -x, q^T x)_+\|_{\beta}}{\|(-q)_+\|_{\beta}}. \quad (26)$$

The number

$$v_{\beta} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} := u_{\beta\beta} \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \left\| \begin{pmatrix} -M \\ -I \\ q^T \end{pmatrix} \right\|_{\beta} \quad (27)$$

is defined as the condition number of the linear program (20).

We note that because $Mx + q \geq 0$, $x \geq 0$ imply that $q^T x \geq 0$, it follows that there exists no x satisfying $Mx > 0$, $x > 0$ and $q^T x < 0$ and hence the problem (24) cannot be reduced to a linear program as was done in Theorem 3.

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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A new explicit bound is given for the ratio of the absolute error in an approximate solution of a system of linear inequalities and equalities to the absolute residual. This bound generalizes the concept of a norm of the inverse of a nonsingular matrix. With this bound a condition number is defined for a system of linear inequalities and equalities and for linear programs. The condition number gives a bound on the ratio of the relative error of an approximate solution to the relative residual. In the case of a strongly stable system of linear inequalities and equalities the condition number can be computed by means of a | | |